

# DELINQUENT NETWORKS

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## Abstract

Delinquents are embedded in a network of relationships. Each delinquent decides in a non-cooperative way how much delinquency effort he will exert. We characterize the Nash equilibrium and derive an optimal enforcement policy, called the key-player policy. We then extend our characterization of optimal single player network removal to optimal group removal, the key group. We also characterize and derive a policy that targets links rather than players. Finally, we endogenize the network connecting delinquents by allowing players to join the labor market instead of committing delinquent offenses. The key-player policy turns out to be much more complex because it depends on wages and on the structure of the network. (JEL: A14, C72, K42, L14)

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## 1. Introduction

Polls show that people regard crime and delinquency as the number one social problem. As such, identifying the root causes of delinquent activity and designing efficient policies against delinquency are two natural scopes for the economics profession. About thirty years ago, the major breakthrough in the economic analysis of crime was the work of Becker (1968), in which delinquents are rational individuals acting in their own self-interest. In deciding to commit a delinquency, delinquents weigh the expected costs against the expected benefits accruing from this activity. The goal of the criminal justice system is to raise expected costs of delinquency to offenders above the expected benefits. People will commit crimes only so long as they are willing to pay the prices society charges.

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There is by now a large literature on the economics of crime. Both theoretical and empirical approaches have been developed over the years in order to better understand the costs and benefits of crime (see, for instance, the literature surveys by Garoupa (1997) and Polinsky and Shavell (2000)). In particular, the interaction between the “delinquency market” and the other markets has important general equilibrium effects that are crucial if one wants to implement the most effective policies.<sup>1</sup> The standard policy tool to reduce aggregate delinquency that is common to all these models relies on the deterrence effects of punishment, that is, the planner should increase, uniformly, punishment costs.

It is, however, well established that delinquency is, to some extent, a group phenomenon, and the source of crime and delinquency is located in the intimate social networks of individuals (see, e.g., Sutherland 1947, Sarnecki 2001 and Warr 2002). Indeed, delinquents often have friends who have themselves committed several offenses, and social ties among delinquents are seen as a means whereby individuals exert an influence over one another to commit crimes.<sup>2</sup> In fact, not only friends but also the *structure* of social networks matters in explaining an individual’s own delinquent behavior. In adolescents’ friendship networks, Haynie (2001) and Calvó-Armengol, Patacchini, and Zenou (2005, 2009) show that individual Bonacich centrality (a standard measure of network centrality) together with the density of friendship links condition the delinquency–peer association. This suggests that the underlying structural properties of friendship networks must be taken into account to better understand the impact of peer influence on delinquent behavior and to address adequate and novel delinquency-reducing policies.

In this paper, we develop an explicit delinquent network game<sup>3</sup> where individuals decide non-cooperatively their crime effort by using the network model developed by Ballester, Calvó-Armengol, and Zenou (2006) to the case of delinquent networks. For this purpose, we build on the Beckerian incentives approach to delinquency behavior but let the cost to commit delinquent offenses to be determined, in part, by one’s network of delinquent mates. We then consider different policies that aim at reducing the total crime activity in a delinquent network. Compared to Ballester, Calvó-Armengol, and Zenou, the present paper has the following innovations: (i) the payoff function contains a component with global strategic substitutes and is parameterized so that the effects of different parameters can be easily interpreted; (ii) it compares the effects of an increase in

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1. For example, Burdett, Lagos, and Wright (2003) study the interaction between crime and unemployment, and Verdier and Zenou (2004) analyze the impact of the land market on criminal activities. Most of these models generate multiple equilibria that can explain why identical areas may end up with different amounts of crime.

2. The empirical evidence collected so far in the economics literature suggests that peer effects are, indeed, quite strong in criminal decisions. See, for instance, Glaeser, Sacerdote, and Scheinkman (1996), Ludwig, Duncan, and Hirschfield (2001), Patacchini and Zenou (2008), Damm and Dustmann (2008), and Bayer, Hjalmarsson, and Pozen (2009).

3. For recent literature surveys on networks, see Goyal (2007) and Jackson (2008).

“punishment” and other standard crime policies with a “key player” policy; (iii) it analyzes a “key group” and a “key link” policy, in addition to the “key player” policy; (iv) it shows that finding a “key group” is an NP-hard problem and provides a simple (greedy) algorithm and a bound for the degree of suboptimality of the algorithm’s solution; (v) it characterizes the equilibrium of a so-called “entry game” where individuals decide whether to continue participating in a delinquent network (in their previously allotted position) or take some job in the outside world; and (vi) it analyzes the “key player” and “key group” policies in the “entry game.” To the best of our knowledge, this is the first paper analyzing policies aiming at reducing crime in an explicit social network framework.<sup>4</sup>

## 2. Delinquency Network Outcomes

### 2.1. The Delinquency Network Game

*The network.* A network  $g$  is a set of ex ante identical individuals  $N = \{1, \dots, n\}$  and a set of links between them. We assume  $n \geq 2$ . The  $n$ -square adjacency matrix  $\mathbf{G}$  of a network  $g$  keeps track of the direct connections in this network. By definition, players  $i$  and  $j$  are directly connected in  $g$  if and only if  $g_{ij} = 1$  (denoted by link  $ij$ ), and  $g_{ij} = 0$  otherwise. Links need not be reciprocal, so that we may have  $g_{12} = 1$  and  $g_{21} = 0$ . Only in some of our results will we explicitly impose this symmetry. By convention,  $g_{ii} = 0$ . Thus  $\mathbf{G}$  is  $(0, 1)$ -matrix with zeros on its diagonal.

*The delinquency decision game.* We focus on petty crimes so we consider delinquents rather than criminals.<sup>5</sup> Consider some delinquency network  $g$ . Delinquents in the network decide how much effort to exert. We denote by  $x_i$  the delinquency effort level of delinquent  $i$ , and by  $\mathbf{x} = (x_1, \dots, x_n)$  the population delinquency profile.

Following Becker (1968), we assume that delinquents trade off the costs and benefits of delinquent activities to make their delinquency effort decision. The expected delinquency gains to delinquent  $i$  are given by

$$u_i(\mathbf{x}, g) = \underbrace{y_i(\mathbf{x})}_{\text{proceeds}} - \underbrace{p_i(\mathbf{x}, g)}_{\text{apprehension}} \underbrace{f}_{\text{fine}}. \quad (1)$$

The individual proceeds  $y_i(\mathbf{x})$  correspond to the gross delinquency payoffs of delinquent  $i$ . Individual  $i$  gross payoff positively depends on  $i$ 's delinquency

4. Calvó-Armengol and Zenou (2004) proposed a network model of criminal activities but without looking at policy issues.

5. It is well documented that social interactions and peer effects are stronger for petty crimes than for other types of crimes (Glaeser, Sacerdote, and Scheinkman 1996; Jacob and Lefgren 2003; Patacchini and Zenou 2008).

involvement  $x_i$ , and on the whole population delinquency effort  $\mathbf{x}$ . The proceeds  $y_i(\mathbf{x})$  indicate the *global*<sup>6</sup> payoff interdependence. The cost of committing delinquency  $p_i(\mathbf{x}, g)$  is also positively related to  $x_i$  as the apprehension probability increases with one's involvement in delinquency, hitherto, with one's exposure to deterrence. In words,  $p_i(\mathbf{x}, g)$  reflects *local* complementarities in delinquency efforts across delinquents directly connected through  $g$ .

The crucial assumption made here is that delinquents improve illegal practice while interacting with their direct delinquent mates. In other words, we assume that the higher the criminal connections to a criminal and/or the higher the involvement in criminal activities of these connections, the lower his individual probability to be caught  $p_i(\mathbf{x}, g)$ . The idea is as follows. *There is no formal way of learning to become a criminal*, no proper "school" providing an organized transmission of the objective skills needed to undertake successful criminal activities. Given this lack of formal institutional arrangement, we believe that the most natural and efficient way to learn to become a criminal is through the interaction with other criminals. Delinquents learn from other criminals belonging to the same network how to commit crime in a more efficient way by sharing the know-how about the "technology" of crime. In our model, we capture this local nature of the mechanism through which skills are acquired by relating the individual probability to be caught to the crime level involvement of one's direct mates, and by assuming that this probability decreases with the corresponding local aggregate level of crime.

This view of criminal networks and the role of peers in learning the technology of crime is not new, at least in the criminology literature. In his very influential theory of differential association, Sutherland (1947) locates the source of crime and delinquency in the intimate social networks of individuals. Emphasizing that criminal behavior is *learned* behavior, Sutherland argued that persons who are selectively or differentially exposed to delinquent associates are likely to acquire that trait as well.<sup>7</sup> In particular, one of his main propositions states that when criminal behavior is learned, the learning includes (i) techniques of committing the crime, which are sometimes very complicated, sometimes very simple, and (ii) the specific direction of motives, drives, rationalization, and attitudes. Interestingly, the positive correlation between self-reported delinquency and the number of delinquent friends reported by adolescents has proven to be among the strongest and one of the most consistently reported findings in the delinquency literature (for surveys, see Warr 1996 and Matsueda and Anderson 1998). In our model, individuals learn illegal conduct from others but practice it alone.

Because in most of the papers cited (from the sociology and criminology literatures), selection and endogeneity issues are not properly addressed, we would

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6. That is, across all criminals in the network.

7. Sutherland (1947) and Akers (1998) expressly argue that criminal behavior is learned from others in the same way that all human behavior is learned. Indeed, young people may be influenced by their peers in all categories of behavior—music, speech, dress, sports, and delinquency.

like to provide some evidence on learning in crime from the economics literature where these econometric issues are taken into account. Damm and Dustmann (2008) investigate the following question: Does growing up in a neighborhood in which a relatively high share of youth has committed crime increase the individual's probability of committing crime later on? To answer this question, Damm and Dustmann exploit a Danish natural experiment that randomly allocates parents of young children to neighborhoods with different shares of youth criminals. With area fixed effects, their key results are that one standard deviation increase in the share of youth criminals in the municipality of initial assignment increases the probability of being charged with an offense within the ages of 18–21 by 8 percentage points (or 23%) for men. This neighborhood crime effect is mainly driven by property crime. Bayer, Hjalmarsson, and Pozen (2009) consider the influence that juvenile offenders serving time in the same correctional facility have on each other's subsequent criminal behavior. They also find strong evidence of learning effects in criminal activities since exposure to peers with a history of committing a particular crime increases the probability that an individual who has already committed the same type of crime recidivates with that crime.

There are clearly learning effects in crime. One may, however, argue that although delinquents learn a lot from their best friends, the learning is not infinite; so that after some time, friends do not provide any positive externalities that may reduce the probability to be caught. This is clearly not true at least for delinquent friendships. Indeed, the techniques and information about crime are not static and are constantly evolving. For example, friends may help a delinquent to be more efficient in shoplifting by explaining the new type of protection that has been installed in a particular shop. This information may not be valid a year later if the shop has changed its protection system. Friends can also tell a delinquent which apartment has already been robbed so that this delinquent can avoid taking a risk without getting many proceeds. Another interesting example is people selling illegal DVDs on the street. They share knowledge based on experience (level of activity) but they decide on their effort separately. To summarize, delinquents who know each other can share information because they are friends (we take this communication as given) even though they act separately in the delinquent world.

For the sake of tractability, we restrict to the following simple expressions:

$$\begin{aligned}
 y_i(\mathbf{x}) &= x_i \max \left\{ A - \delta \sum_{j=1}^n x_j, 0 \right\}, \\
 p_i(\mathbf{x}, g) &= p_0 x_i \max \left\{ 1 - \phi \sum_{j=1}^n g_{ij} x_j, 0 \right\},
 \end{aligned} \tag{2}$$

where  $A > 0$ ,  $\delta > 0$  and  $\phi \geq 0$ . For the sake of simplicity, we take  $A = 1$ . We assume that, at an equilibrium  $\mathbf{x}^*$ ,<sup>8</sup>

$$1 - \delta \sum_{j=1}^n x_j^* \geq 0 \quad \text{and} \quad 1 - \phi \sum_{j=1}^n g_{ij} x_j^* \geq 0.$$

Then, by direct substitution, we can focus on the following utility function:

$$u_i(\mathbf{x}, g) = (1 - \pi)x_i - \delta x_i^2 - \delta \sum_{j \neq i} x_i x_j + \pi \phi \sum_{j=1}^n g_{ij} x_i x_j, \quad (3)$$

where  $\pi = p_0 f$  is the marginal expected punishment cost for an isolated delinquent. We assume throughout that  $\pi < 1$ . With these expression, we have

$$\sigma_{ij} = \frac{\partial^2 u_i(\mathbf{x}, g)}{\partial x_i \partial x_j} = \begin{cases} \bar{\sigma} = -\delta + \pi \phi g_{ij} & \text{if } g_{ij} = 1, \\ \underline{\sigma} = -\delta & \text{if } g_{ij} = 0, \end{cases} \quad (4)$$

so that  $\sigma_{ij} \in \{\underline{\sigma}, \bar{\sigma}\}$ , for all  $i \neq j$  with  $\underline{\sigma} \leq 0$ . The parameter  $\delta \geq 0$  measures the intensity of the global interdependence on gross delinquency pay-offs. Here, individual delinquency efforts are global strategic substitutes. The optimal delinquency effort of a given delinquent thus decreases with the delinquency involvement of any other delinquent in the network. The expression  $\pi \phi > 0$  captures the local strategic complementarity of efforts on the apprehension probability. This expression is non-zero only when  $g_{ij} = 1$ , that is, when delinquents  $i$  and  $j$  are directly linked to each other. Finally, note that  $\partial^2 u_i(\mathbf{x}, g) / \partial x_i^2 = -2\delta < 0$ .

## 2.2. Nash Equilibrium

Let  $\mathbf{G}^k$  be the  $k$ th power of  $\mathbf{G}$ , with coefficients  $g_{ij}^{[k]}$ , where  $k$  is some integer. The matrix  $\mathbf{G}^k$  keeps track of the indirect connections in the network:  $g_{ij}^{[k]} \geq 0$  measures the number of walks of length  $k \geq 1$  in  $g$  from  $i$  to  $j$ . In particular,  $\mathbf{G}^0 = \mathbf{I}$ . Given a scalar  $a \geq 0$  and a network  $g$ , we define the following matrix:

$$\mathbf{M}(g, a) = [\mathbf{I} - a\mathbf{G}]^{-1} = \sum_{k=0}^{+\infty} a^k \mathbf{G}^k.$$

8. This assumption is satisfied, for instance, when  $\delta \geq \phi$  or  $\pi \geq 1/2$ .

We define the vector of Katz–Bonacich<sup>9</sup> centralities of parameter  $a$  in  $g$  as follows:

$$\mathbf{b}(g, a) = [\mathbf{I} - a\mathbf{G}]^{-1} \cdot \mathbf{1}, \quad (5)$$

where  $\mathbf{1}$  is the  $n$ -dimensional vector of ones. For all  $\mathbf{y} \in \mathbb{R}^n$ ,  $y = y_1 + \dots + y_n$  is the sum of its coordinates. Define  $b(g, \theta) = \sum_{i=1}^{i=n} b_i(g, \theta)$ , denote  $\theta = \pi\phi/\delta$  and let  $\rho(g)$  be the *spectral radius* of the adjacency matrix  $\mathbf{G}$ . We have the following result:

**PROPOSITION 1.** (Ballester, Calvó-Armengol, and Zenou 2006) *If  $\theta\rho(g) < 1$ , then there exists a unique Nash equilibrium  $\mathbf{x}^*$ , which is interior, and given by*

$$\mathbf{x}^* = \frac{1 - \pi}{\delta[1 + b(g, \theta)]} \mathbf{b}(g, \theta). \quad (6)$$

The equilibrium Katz–Bonacich centrality measure  $\mathbf{b}(g, \theta)$  is thus the relevant network characteristic that shapes equilibrium behavior. The condition  $\theta\rho(g) < 1$  relates the payoff function to the network topology and guarantees that local complementarities are not too large compared to own concavity. When this condition holds,<sup>10</sup> (and only then), the matrix  $[\mathbf{I} - \theta\mathbf{G}]^{-1}$  can be developed into the infinite sum  $\sum_{k \geq 0} \theta^k \mathbf{G}^k$ , which brings the Katz–Bonacich centrality measure into the picture.

### 2.3. Comparative Statics

In Proposition 1, the individual and aggregate delinquency levels depend on the underlying network  $g$  connecting them through the adjacency matrix  $\mathbf{G}$  in equation (6). The next result establishes a positive relationship between the equilibrium aggregate delinquency level and the network pattern of connections.

We write  $g \subset g'$  to denote that the set of links in  $g'$  contains the links in  $g$ , that is, for all  $i, j$ ,  $g'_{ij} = 1$  if  $g_{ij} = 1$ .

**PROPOSITION 2.** (Ballester, Calvó-Armengol, and Zenou 2006) *Let  $g$  and  $g'$  be symmetric networks such that  $g \subset g'$ . If  $\theta\rho(g') < 1$ , then, in equilibrium, the total delinquency level under  $g'$  is strictly higher than that under  $g$ .*

Consider two nested networks  $g$  and  $g'$  such that  $g \subset g'$ . Then, either  $g$  and  $g'$  connect the same number of delinquents but there are more direct links

9. Due to Katz (1953) and Bonacich (1987).

10. Testing the impact of the Katz–Bonacich centrality measure on educational and criminal outcomes in the United States, Calvó-Armengol, Patacchini, and Zenou (2005, 2009) found that only 18 out of 199 networks (i.e., 9%) do not satisfy this condition.

between them in  $g'$  than in  $g$ , or  $g'$  brings additional individuals into the pool of delinquents already connected by  $g$ , or both. Proposition 2 shows that the density of network links and the network size (or boundaries) affect positively aggregate delinquency, a feature often referred to as the *social multiplier effect*.<sup>11</sup>

### 3. Delinquency Network Policies

#### 3.1. Finding the Key Player

The standard policy tool to reduce aggregate delinquency relies on the deterrence effects of punishment (see for example Becker 1968). Formally, an increase in  $\pi$ , which translates into an increase in  $\theta$ , amounts to hardening punishment costs borne by delinquents. Our previous results associate a distribution of delinquency efforts across delinquents to any delinquency network connecting them. In this case, an increase in  $\theta$  affects all delinquency decisions simultaneously and shifts the whole delinquency efforts distribution to the left, thus reducing the average (and the aggregate) delinquency level. In our model, though, delinquent behavior is tightly rooted in the network structure. When all delinquents hold homogeneous positions in the delinquency network, they all exert a similar delinquency effort. In this case, the previously mentioned policy, which tackles average behavior and does not discriminate among delinquents depending on their relative contribution to the aggregate delinquency level, may be appropriate. However, if delinquents hold very heterogeneous positions in the delinquency network, they contribute very differently to the aggregate delinquency level. The variance of efforts is higher. In this case, we could expect a sharp reduction in average delinquency by directly removing a delinquent from the network and thus altering the whole distribution of delinquency efforts, not just shifting it. A targeted policy that discriminates among delinquents depending on their location in the network may then be more appropriate. The key player is the one inducing the highest aggregate delinquency reduction. Formally, the planner's problem is the following:

$$\max\{x^*(g) - x^*(g_{-i}) \mid i = 1, \dots, n\},$$

which, when the original delinquency network  $g$  is fixed, is equivalent to:

$$\min\{x^*(g_{-i}) \mid i = 1, \dots, n\} \quad (7)$$

From Ballester, Calvó-Armengol, and Zenou (2006), we now define a new network centrality measure  $d(g, \theta)$  that will happen to solve this compromise.

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11. See, for instance, Glaeser, Sacerdote, and Scheinkman (2003), and references therein.



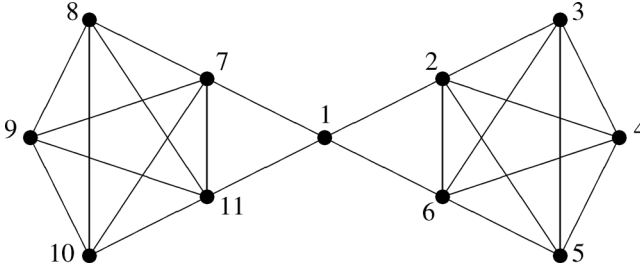


FIGURE 1. Bridge network with eleven delinquents.

DEFINITION 1. For all network  $g$  and for all  $i$ , the measure

$$d_i(g, \theta) = b(g, \theta) - b(g_{-i}, \theta) = \frac{b_i(g, \theta)^2}{m_{ii}(g, \theta)} \tag{8}$$

accounts for the number of walks that crosses player  $i$  in the network  $g$ .

The intercentrality measure  $d_i(g, \theta)$  of delinquent  $i$  is the sum of  $i$ 's centrality measures in  $g$ , and  $i$ 's contribution to the centrality measure of every other delinquent  $j \neq i$  also in  $g$ . It accounts both for one's exposure to the rest of the group and for one's contribution to every other exposure.

PROPOSITION 3. (Ballester, Calvó-Armengol, and Zenou 2006) *A player  $i^*$  is the key player that solves equation (7) if and only if  $i^*$  is a delinquent with the highest intercentrality in  $g$ , that is,  $d_{i^*}(g, \theta) \geq d_i(g, \theta)$ , for all  $i = 1, \dots, n$ .*

Observe that the key player policy is such that the planner perturbs the network by removing a delinquent and all other delinquents are allowed to change their effort after the removal but the network is not “rewired,” that is, individuals do not optimally change their relationships (links) with their friends. This assumption can be justified for two reasons. First, it would be extremely difficult to solve a network formation problem every time a player is removed. Second, in the context of a short-term policy and because friendship relationships take longer to adjust than the level of criminal activity, it is reasonable to assume that delinquents do not change their friends when one of them is removed even though they can modify their crime activity.

EXAMPLE 1. Consider the network  $g$  in Figure 1 above.

TABLE 1. Bonacich centrality versus key player.

$\theta$	0.1			0.2		
	$x_i$	$b_i$	$d_i$	$x_i$	$b_i$	$d_i$
1	0.077	1.75	2.92	0.072	8.33	41.67*
2	0.082*	1.88*	3.28*	0.079*	9.17*	40.33
3	0.075	1.72	2.79	0.067	7.78	32.67

\* = column highest value.

We distinguish three different types of equivalent players in this network: Type-1 player, which is player 1; type-2 players, which encompass players 2, 6, 7, and 11, and type-3 players, which are players 3, 4, 5, 8, 9, and 10. Table 1 computes, for delinquents of types 1, 2, and 3, the value of delinquency efforts  $x_i$ , centrality measures  $b_i(g, \theta)$ , and intercentrality measures  $d_i(g, \theta)$  for different values of  $\theta$  and with  $\delta = \phi = 1$ . In each column, a variable with a star identifies the highest value.<sup>12</sup>

First note that type-2 delinquents always display the highest  $b$ -centrality measure. These delinquents have the highest number of direct connections. Besides, they are directly connected to the bridge delinquent 1, which gives them access to a very wide and diversified span of indirect connections. Altogether, they are the most  $b$ -central delinquents. For low values of  $\theta$ , the direct effect on delinquency reduction prevails, and type-2 delinquents are the key players—those with highest intercentrality measure  $d_i$ . When  $\theta$  is higher, though, the most active delinquents are no longer the key players. Now, indirect effects matter a lot, and eliminating delinquent 1 has the highest joint direct and indirect effect on aggregate delinquency reduction.

### 3.2. Comparing Policies

*3.2.1. The Cost of Finding the Key Player.* Given a delinquency network  $g$  and a punishment cost  $\theta$ , the ranking of delinquents according to their individual intercentrality measure  $d_i(g, \theta)$  provides a criterion for the selection of an optimal target in the network. Implementing such a network-based policy obviously has its costs. Indeed, the computation of the intercentrality measures relies on the knowledge of the adjacency matrix of the delinquency network. This matrix is obtained from sociometric data that identify the network links between delinquents. It is important to note that sociometric data on delinquency are available in many cases. For instance, Haynie (2001) and Calvó-Armengol, Patacchini, and Zenou (2005) use friendship data to identify delinquent peer networks for adolescents in 134 schools in the U.S. that participated in an in-school survey in

12. We can compute the highest possible value for  $\theta$  compatible with our definition of centrality measures, equal to  $\hat{\theta} = 2/(3 + \sqrt{41}) \simeq 0.213$ .

the 1990s. Sarnecki (2001) provides a comprehensive study of co-offending relations and corresponding network structure for football hooligans and right-wing extremists in Stockholm. Baker and Faulkner (1993) reconstruct the structure of conspiracy networks for three well-known cases of collusion in the heavy electrical equipment industry in the U.S. In all these cases, one may directly use the available data to compute the intercentrality measures.

In some other cases, though, ad hoc information gathering programs have to be implemented. Interestingly, Costebander and Valente (2003) show that centrality measures based on connectivity (rather than betweenness), such as  $\mathbf{b}$  and  $d$ , are robust to misspecifications in sociometric data, and thus open the door to estimations of centrality measures with incomplete samples of network data. This, obviously, reduces the cost of identifying the key player. The idea behind these results is that these measures take into account all walks in the network. Thus, generally the centrality of a player is not determined only by his direct links but by the complete structure of the network. In this sense, the probability that a missing link affects the choice of the most central/intercentral player is smaller than with other types of measures. This difference turns significant the higher the value of the density parameter  $\theta$  since, in that case, higher order walks are also taken into account in computing the centrality/intercentrality of a player.

*3.2.2. Key Player Versus Random Target.* To fully assess the relevance of the key player delinquency policy, we also need to evaluate the relative returns from following this network targeted policy. For this purpose, we compare the reduction in aggregate delinquency following the elimination of the key player with respect to the expected consequences when the target is selected randomly. For each delinquent  $i$  in the delinquency network, define

$$\eta_i(g) = n \frac{x^*(g) - x^*(g_{-i})}{\sum_{j=1}^n [x^*(g) - x^*(g_{-j})]}.$$

This is the ratio of returns (in delinquency reduction) when  $i$  is the selected target versus a random selection with uniform probability for all delinquents in the network.

Denote by  $\bar{d}(g, \theta)$  the average of the intercentrality measures in network  $g$ , and by  $\sigma_d(g, \theta)$  the standard deviation of the distribution of these intercentrality measures. The following result establishes a lower bound on the ratio of returns in delinquency reduction when the key player is removed.

**PROPOSITION 4.** *Let  $i^*$  be the key player in  $g$  for a given  $\theta$ . Then,*

$$\eta_{i^*}(g) \geq 1 + \frac{\sigma_d(g, \theta)}{\bar{d}(g, \theta)}.$$

See Ballester, Calvó-Armengol, and Zenou (2009) for the proof.

The relative gains from targeting the key player instead of operating a selection at random in the delinquency network increase with the variability in intercentrality measures across delinquents as captured by  $\sigma_d(g, \theta)$ . In other words, the key player prescription is particularly well-suited for networks that display stark location asymmetries across nodes. In these cases, it is more likely that the relative gains from implementing such a policy compensate for its relative costs.

**3.2.3. Key Player Versus Standard Deterrence Policy.** Consider the key player removal policy. When a delinquent is removed from the network, the intercentrality measures of all the delinquents that remain active are reduced, that is,  $d_j(g_{-i^*}, \theta) \leq d_j(g, \theta)$ , for all  $j \neq i^*$ , which triggers a decrease in delinquency involvement for all of them. Moreover, when delinquent  $i^*$  is removed from the delinquency network, the corresponding ratio of aggregate delinquency reduction with respect to the network centrality reduction is an increasing function of the intercentrality measure  $d_i(g, \theta)$  of this delinquent. Formally,

$$\frac{\partial}{\partial d_i(g, \theta)} \left[ \frac{x^*(g) - x^*(g_{-i})}{b(g, \theta) - b(g_{-i}, \theta)} \right] > 0.$$

In words, the target policy displays amplifying effects, and the gains following the judicious choice of the key player (the one with highest intercentrality measure) go beyond the differences in intercentrality measures between this player and any other delinquent in the network.

Consider standard deterrence (“uniform”) policies that consist in increases in  $\theta$ . In particular, consider policies increasing  $\pi$  (i.e. increase in the fine  $f$ ), or reducing  $\delta$ , or increasing  $\phi$ .

Observe first that an increase of  $\pi$  above 1 would induce an equilibrium with no delinquency. The problem is that the condition  $\rho(g)\pi\phi/\delta < 1$  in Proposition 1 that guarantees the existence of a unique interior Nash equilibrium and that the Bonacich centrality measure is well defined may no longer be satisfied. Moreover, we are interested in situations where it is costly for the authorities to increase  $\pi$  (or to implement any other policy). A thorough analysis of how the costs of different policies affect the choice of the “right” policy is, however, beyond the scope of this paper.

Let us now focus on the effect of  $\pi$ ,  $\delta$ , or  $\phi$  on  $x^* = \mathbf{1}^T \mathbf{x}^*$ , the equilibrium aggregate delinquency activity. Observe that we are dealing with the situation of a unique equilibrium under Proposition 1. From expression (6), it is easy to obtain

$$x^* = \frac{(1 - \pi)b(g, \theta)}{\delta[1 + b(g, \theta)]}. \quad (9)$$

It is then straightforward to show that the aggregate delinquency activity  $x^*$  is increasing in the local complementarity parameter  $\phi$  but is decreasing in the

global substitutability parameter  $\delta$ . However, the effect of  $\pi$  on  $x^*$  is ambiguous and given by

$$\frac{\partial x^*}{\partial \pi} = \underbrace{-\frac{x^*}{1-\pi}}_{\text{direct negative effect}} + \underbrace{\frac{\phi}{\delta} \frac{\partial x^*}{\partial \theta}}_{\text{indirect positive effect}},$$

where

$$\frac{\partial x^*}{\partial \theta} = \frac{(1-\pi)}{\delta} \frac{1}{[1+b(g,\theta)]^2} \frac{\partial b(g,\theta)}{\partial \theta} > 0.$$

The impact on punishment results from the combination of two effects that work in opposite directions. First, the individual probability to be apprehended, and thus the punishment costs borne by each delinquent, increases with  $\pi$ . This is a direct negative effect. Second, when  $\pi$  increases, delinquents react strategically by acquiring a better delinquency technology to thwart the higher deterrence they now face. The improvement in delinquency technology stems from more intense know-how inflows and transfers in the delinquency network. Each delinquent centrality measure  $b_i(g, \theta)$  increases, which translates into a higher delinquency involvement for each delinquent. This is an indirect positive effect on aggregate delinquency that mitigates the direct negative effect. In order to better understand this last effect, we run numerical simulations in Ballester, Calvó-Armengol, and Zenou (2009) for  $\delta = 0.1$  and for which the maximum value of  $\pi$  is consistent with the spectral condition of Proposition 1. We consider two cases:  $\phi = 0.8$  and  $\phi = 0.1$ . When the network is more connected ( $\phi = 0.8$ ), namely, both direct and indirect links matter very much, increasing punishment  $\pi$  increases total delinquent activity because the indirect positive effect dominates the direct negative effect. When  $\phi = 0.1$ , meaning that friends of friends have not that much influence on delinquents, then the network effect becomes unimportant compared to the deterrence effect and an increase in punishment  $\pi$  reduces total delinquency  $x^*$ . These results imply that the policymaker should be aware of the degree of connectivity of the network if it is to implement a deterrence policy aiming at reducing delinquent activity. In particular, if a network of delinquents is dense and well connected so that  $\phi$  is high, it should be clear that a key-player policy will be more effective in reducing total delinquency than an increase in punishment.

### 3.3. From the Key Player to the Key Group

So far, we have characterized *optimal single player removal* from the network to reduce delinquency, a *key player*. We now characterize *optimal group removal* from the network, a *key group*.

*3.3.1. Finding the Key Group of Players.* We wish to eliminate a group of  $s$  players from the current population. If we remove a set  $S$  of players such that  $|S| = s$ , the network of delinquents becomes  $g_{-S}$ . The problem is therefore to minimize  $x^*(g_{-S})$  by picking the adequate set  $S$  from the population. Formally, the planner maximizes the total change in delinquent activity:

$$\max_{|S| \leq s} x^*(g) - x^*(g_{-S}),$$

which is equivalent to

$$\min_{|S| \leq s} x^*(g_{-S}). \quad (10)$$

This is a finite optimization problem, which admits at least one solution. Let  $S^*$  be a solution to expression (10). We call the set  $S^*$  a *key group* of the game. Removing  $S^*$  from the game has the highest overall impact on delinquency.

In the following, we assume that the condition on eigenvalue in Proposition 1 holds in the game, guaranteeing the uniqueness of Katz–Bonacich solutions in any game induced by a subset of players. The reason is that  $\rho(g) \geq \rho(g_{-S})$ . As a consequence, if  $\mathbf{b}(g, \theta)$  is well-defined and non-negative (as implied by the condition in Proposition 1), so is  $\mathbf{b}(g_{-S}, \theta)$ .

**DEFINITION 2.** The group intercentrality of  $S$  in the network  $g$  is

$$d_S(g, \theta) = b(g, \theta) - b(g_{-S}, \theta).$$

In fact,  $d_S(g, \theta)$  is the weighted number of walks in  $g$  crossing some agent in  $S$ . The case  $s = 1$  obviously corresponds to the case of finding the key player in the delinquency network. As with the key player, it turns out that diminishing the aggregate delinquent activity reduces to choose the set with the highest group intercentrality:

$$\max_{|S|=s} d_S(g, \theta), \quad (11)$$

that is, the solution<sup>13</sup> of (11) is  $S^* \subseteq N$  such that  $d_{S^*}(g, \theta) \geq d_S(g, \theta)$ , for all  $S \subseteq N$  with  $|S| = s$ .

**REMARK 1.** An equivalent formulation of the key group problem (11) is

$$\max_{\{i_1, \dots, i_s\} \subseteq N} d_{i_1}(g, \theta) + d_{i_2}(g_{-\{i_1\}}, \theta) + d_{i_3}(g_{-\{i_1, i_2\}}, \theta) + \dots + d_{i_s}(g_{-\{i_1, \dots, i_{s-1}\}}, \theta), \quad (12)$$

where  $i_1, \dots, i_s$  are different two by two.

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13. Note that we restrict the maximization program to  $|S| = s$ , given that  $d_S(g, \theta)$  is obviously increasing in the size of  $S$ .

TABLE 2. Key groups of size 2.

Removed Group $S$	$d_S(g, \theta)$
$\{2, 7\}^*$	67.22
$\{2, 8\}$	64.01
$\{3, 8\}$	59.39
$\{1, 2\}$	56.66
$\{2, 6\}$	50.41
$\{2, 3\}$	46.96
$\{3, 4\}$	42.15

\* = column highest value.

In words, the key group maximizes the sum of the individual intercentrality measures of its members across the networks obtained through sequential removal of these members.<sup>14</sup> The idea behind this expression is as follows. We must eliminate a set of players  $S = \{i_1, \dots, i_s\}$  in order to minimize the total number of weighted walks in the network,  $b(g_{-S}, \theta)$ . After deleting player  $i_1$ , the resulting number of walks is  $b(g, \theta) - d_{i_1}(g, \theta)$ . Now, the expression  $d_{i_2}(g_{-i_1}, \theta)$  counts the number of walks that hit agent  $i_2$  once agent  $i_1$  has been eliminated, so that we are not counting the same walk twice. Thus,  $b(g, \theta) - d_{i_1}(g, \theta) - d_{i_2}(g_{-i_1}, \theta)$  is the remaining set of walks after eliminating players  $i_1$  and  $i_2$ , keeping in mind that we only want to count each walk once. By the previous argument, also note that the remaining set of weighted walks is the same if we change the order of deletion of these two players, that is:

$$b(g, \theta) - d_{i_1}(g, \theta) - d_{i_2}(g_{-i_1}, \theta) = b(g, \theta) - d_{i_2}(g, \theta) - d_{i_1}(g_{-i_2}, \theta).$$

Extending this argument to the rest of the players in  $S$ , we obtain expression (12).

EXAMPLE 2. Consider again the network  $g$  in Figure 1 with 11 delinquents and a decay factor  $\theta = 0.2$  and consider the case where the required group size is  $s = 2$ . Table 2 shows the values of *group intercentrality*  $d_S(g, \theta)$  for each possible subset  $S$  of size 2 when  $\theta = 0.2$ . For the sake of simplicity, subsets that yield the same network architecture when they are removed are considered as equivalent.

The key group is  $\{2, 7\}$ , that is, a set of two maximally connected nodes (with five direct contacts each), both connected to the intercentral player 1, and each at a different side of this player. This subset solves the following optimization problem:

$$\max_{i,j} d_{\{i,j\}}(g, \theta) = \max_{i,j} (d_i(g, \theta) + d_j(g_{-i}, \theta)).$$

14. Note that this sum is independent of the order in which nodes are removed.

Suppose that we were to approximate the solution to this optimization problem with a greedy heuristic procedure that sequentially picks up the player that maximizes the individual intercentrality at each step. Formally, let

$$i_1^* = \arg \max_{i \in N} d_i(g, \theta)$$

and then, at each step  $2 \leq t \leq s$ , choose the player  $i_t^*$  with maximum intercentrality in the network where the previous players have been deleted, that is,

$$i_t^* \in \arg \max \{d_i(g_{-\{i_1^*, \dots, i_{t-1}^*\}}, \theta) : i \in N \setminus \{i_1^*, \dots, i_{t-1}^*\}\}$$

breaking possible ties arbitrarily. This greedy algorithm first eliminates player 1, and then any other remaining player (after player 1 has been removed, all the other players have identical positions in the network). Thus, the algorithm returns a group which is not optimal: There are other groups that are better candidates than  $\{1, 2\}$ . Indeed, in this example, player 1 is not only very intercentral, but also his intercentrality is very much correlated with the intercentrality of others. Hence, being greedy and eliminating it at the first stage reduces the chance of finding highly central players at further stages. And, in fact, player 1 is not part of the key group!

Nevertheless, we have obtained a relatively accurate approximation for the result by a simple greedy algorithm, instead of choosing among all possible pairs of agents. Note that, in this example, the error of this approximation is

$$\frac{d_{\{2,7\}}(g, \theta) - d_{\{1,2\}}(g, \theta)}{d_{\{2,7\}}(g, \theta)} \approx 16\%.$$

In fact, when  $s = 2$ , this error can be at most 25%. In the next section devoted to algorithmic considerations, we discuss this issue more generally.

*3.3.2. Algorithmic Considerations.* We prove that the key group problem has an inherent complexity that suggests the use of approximation algorithms. In particular, we study the performance of a greedy procedure where the optimal group is constructed by iteratively choosing an optimal vertex from the network. For a description of NP-hard problems and properties, see Garey and Johnson (1978) and Ballester (2004).

Now, we show that the key group problem is NP-hard, even when we want to completely disrupt the game. First, note that if we were to implement a “brute-force” basic algorithm to find a key group of  $s$  players, we would have to step over all possible  $\binom{n}{s}$  groups of players and compute each particular contribution to the game. This combinatorial procedure may involve up to an exponential number of steps in  $n$ . The computational complexity here is mainly combinatorial, that is,



although computing the contribution of a given group to the activity of the game is computationally tractable, the fact that this task has to be done an exponential number of times (in the worst case) makes the problem potentially intractable. NP-hardness relates to the difficulty of computationally solving a particular class of problems. Hence, by showing that the key group problem is NP-hard, we show that there is no possible sophisticated algorithm such that, given any network, will return the exact key group in reasonable time.<sup>15</sup> This means that the key group problem is a NP-hard problem, from the combinatorial perspective. Nevertheless, we will show herein that we can efficiently approximate it.

**PROPOSITION 5.** *The problem of finding a key group in a network  $g$  is NP-hard.*

See Ballester, Calvó-Armengol, and Zenou (2009) for the proof.

Because the computational complexity inherent to the key group selection is high, it is suitable to use algorithmic approximations in order to solve real-life problems with large networks.

Consider a *greedy algorithm* that sequentially eliminates in  $s$  stages the player with highest intercentrality, that is, let  $S^G = \{i_1^G, \dots, i_s^G\}$  such that for all  $t = 1, \dots, s$ , player  $i_t^G$  is the most intercentral in  $g_{-S_t}$ , where  $S_t = \{i_1^G, \dots, i_t^G\}$ . We have the following result.

**PROPOSITION 6.** *The key group problem can be approximated in polynomial-time by the use of a greedy algorithm, where, at each step  $t$ , expression (8) is used to find the agent  $i_t^G$  who will become a member of the approximated key group  $S^G$ . The error of the approximation can be bounded as follows.*

$$\varepsilon \equiv \frac{d_{S^*}(g, \theta) - d_{S^G}(g, \theta)}{d_{S^*}(g, \theta)} < \frac{1}{e} \approx 36.79\%.$$

See Ballester, Calvó-Armengol, and Zenou (2009) for the proof.

This proposition shows that the error of approximation of using a greedy algorithm instead of solving the key group problem directly is at most 36.79%.<sup>16</sup> If the approximation error is over 30% in most situations then it would be difficult to claim that this result provides a good approximation. In Ballester, Calvó-Armengol, and Zenou (2009), we provide some numerical simulations for 100 different possible networks and for  $n = 10$  and  $n = 15$ , and show that the relative

15. This fact is conjectured by nearly all computer scientists who believe that there is no such algorithm for solving any NP-hard problem. A simple reason for this is that, after decades of continuous search, no one has found an efficient algorithm for solving any NP-hard problem.

16. As Nemhauser, Wolsey, and Fisher (1978) have showed, the error bound obtained is tight.

error of approximation  $\varepsilon$  is really small, at most 1.7%. Therefore, we are confident that the use of the greedy procedure can be guaranteed to provide a fairly good approximation  $S^G$  for the true solution  $S^*$  of the problem.

### 3.4. Finding the Key Link

Let us now focus on a different crime policy that targets links rather than individuals. The aim of this policy is to choose how to optimally remove a link (or a set of links) between two individuals in order to minimize the total delinquency level. In some situations, the limitation of resources or the nature of the problem requires to optimally choose among the set of dependencies among players. For instance, a social planner would like to optimally reduce the (communication) externalities among delinquents subject to a restriction in the number  $r$  of bilateral influences that can be targeted. This situation can be interpreted as a problem of optimally removing a set of links from the network.

Let us illustrate this policy with real-world examples. As stated previously, a link removal means a disruption of the communication between two delinquents. For instance, when a policeman is watching the street, he is somehow disrupting the possible communication between delinquents from the same neighborhood (a link can be understood as communication in a particular place). This policeman is not, however, avoiding communication with other delinquents somewhere else. Another example is to put a delinquent teenager in another school where there are fewer delinquents. By doing so, this delinquent will stop his activities and communication with other delinquents in the older school. In this section, we will not compare link-removal and player-removal policies because they depend on the costs for the policymaker, and we are not dealing with this issue. The key-link policy should, however, be understood as closely related to the key-player policy because the removal of a player implies the removal of his links plus the removal of the isolated player that remains. Removing a set of links is somewhat more flexible because the policymaker can target links from different players.

More formally, for  $g' \subset g$ , let  $l_{g'}(g, \theta)$  be the number of walks in  $g$  (weighted by  $\theta$ ) that use some edge in  $g'$ . This is the contribution of  $g'$  to the total connectivity of  $g$ .

Suppose that we need to maximize the change in network activity after removing at most  $r$  links. Our best choice will consist of  $r$  links from the set of all possible links not present in  $g$ . Formally, we need to solve

$$\max_{g' \subseteq g} \{l_{g'}(g, \theta) : |g'| \leq r\}.$$

Consider again the bridge network described in Figure 1. As in the case of the key player, even when  $r = 1$ , the optimal choice can depend on the strength of complementarities, as shown in Table 3. For moderate values of  $\theta$

TABLE 3. Removing a link and reduction in Bonacich centralities.

Removed link $\{i, j\}$	Reduction in $b(g, 0.1)$	Reduction in $b(g, 0.22)$
$\{1,2\}$	0.59	185.99*
$\{2,6\}$	0.63*	180.84
$\{2,3\}$	0.58	164.37
$\{3,4\}$	0.53	148.95

\* = column highest value.

(i.e.,  $\theta = 0.1$ ), the key link to be removed is the one between the most central nodes (i.e., delinquents 2 and 6). However, for higher values of  $\theta$  (i.e.,  $\theta = 0.22$ ), intermediate positions become more relevant and the key link is part of the bridge between the two clusters in the network (i.e., delinquents 1 and 2).

Let us now derive more general results. We first deal with the case of directed links (non-symmetric networks) because it provides an easier expression of our result. In this case, the planner has more degrees of freedom because it can target specific directed links. Let  $h \equiv g \setminus \{ij\}$  be the network  $g$  where  $g_{ij}$  is set to zero. The following relation holds in this class of networks, for all pair of agents  $k, l \in N$ :

$$m_{kl}(g, \theta) - m_{kl}(h, \theta) = \theta m_{ki}(h, \theta) m_{jl}(g, \theta). \quad (13)$$

That is, all walks from  $k$  to  $l$  arrive at  $i$  for the first time before crossing  $ij$  (so this set of walks occurs in the network  $h$ ), cross the link  $ij$  and then continue from  $j$  to  $l$  in the network  $g$ . Let  $l_{ij}(g, \theta)$  be the total contribution of link  $ij$  to the centrality of  $g$ :

$$l_{ij}(g, \theta) \equiv \sum_{k,l \in N} (m_{kl}(g, \theta) - m_{kl}(h, \theta)).$$

Let  $\tilde{b}_i(g, \theta)$  be the Katz–Bonacich *in*-centrality of player  $i$ , namely, the weighted sum of the value of walks entering node  $i$  in the network  $g$ :

$$\tilde{b}_i(g, \theta) = \sum_{j=1}^n m_{ji}(g, \theta).$$

LEMMA 1. *The contribution of a single directed link  $ij \in g$  to the total Katz–Bonacich centrality of the network  $g$  is given by*

$$l_{ij}(g, \theta) = \theta \frac{\tilde{b}_i(g, \theta) b_j(g, \theta)}{1 + \theta m_{ji}(g, \theta)}. \quad (14)$$

See Ballester, Calvó-Armengol, and Zenou (2009) for the proof.

This expression reflects the asymmetry of players  $i$  and  $j$  under the assumption of directed links. The effect of a directed link  $ij$  depends roughly on the in-centrality of player  $i$  and the out-centrality of player  $j$ .

When links are *undirected*, the following expression allows us to compute the contribution of a single link  $ij \in g$  to the total Katz–Bonacich centrality of the network  $g$ . The proof is omitted, being similar to the case of directed links.

LEMMA 2. *The contribution of a single undirected link  $ij \in g$  to the total Katz–Bonacich centrality of the network  $g$  is given by*

$$l_{ij}(g, \theta) = \theta(b_i(h, \theta)b_j(g, \theta) + b_i(g, \theta)b_j(h, \theta)) \\ = \theta \frac{2b_i(g, \theta)b_j(g, \theta)(1 + \theta m_{ij}(g, \theta)) - \theta[b_i^2(g, \theta)m_{jj}(g, \theta) + b_j^2(g, \theta)m_{ii}(g, \theta)]}{[1 + \theta m_{ij}(g, \theta)]^2 - \theta^2 m_{ii}(g, \theta)m_{jj}(g, \theta)}. \quad (15)$$

In order to provide an interpretation, we take a moderate  $\theta$ . Then,  $l_{ij}(g, \theta)$  is proportional to  $b_i(g, \theta)b_j(g, \theta)$ . This means that, for moderate values of  $\theta$ , the key link is the one connecting any two nodes with highest Katz–Bonacich centrality. This was the case in the earlier example where the link  $\{2, 6\}$  was chosen when  $\theta = 0.1$ .

Expressions (14) and (15) have obvious advantages, as expression (8) does in the case of the key player. We can compute the contribution of any link  $ij$  from the current data  $\mathbf{M}(g, \theta)$  without having to recompute an inverse  $\mathbf{M}(g \setminus \{ij\}, \theta)$  for each  $ij \in g$ . These operations are clearly cheaper than the computation of an inverse. This fact becomes critical if we are to approximate the optimal interaction set. The reason is that the function  $l_{g'}(g, \theta)$  is submodular in  $g'$  so that we can iteratively find the maximum of  $l_{ij}(g, \theta)$  using equations (14) or (15) to obtain quickly a good approximation of the problem.

PROPOSITION 7. *The key interaction set problem can be approximated in polynomial-time by the use of a greedy algorithm where, at each step, expression (14) or (15) is used to find the link  $ij$  (with highest  $l_{ij}(g, \theta)$ ) that will become a member of the approximated key interaction set. The error of the approximation can be bounded as follows:*

$$\varepsilon \equiv \frac{l_{g^*}(g, \theta) - l_{g^G}(g, \theta)}{l_{g^*}(g, \theta)} < \frac{1}{e} \approx 36.79\%.$$

## 4. Joining Delinquency Networks

### 4.1. Equilibrium Networks

In this section, we extend our game in order to allow individuals to choose whether they want to participate in the crime market or not in the first stage. So far, we have assumed that the delinquency network was given. In some cases, though, delinquents may have opportunities outside the delinquency network. In particular, petty delinquents may consider entering the labor market and giving up delinquent activities. Here, we expand the model and endogenize the delinquency network by allowing delinquents to take a binary decision on whether to stay in the delinquency network or to drop out of it.<sup>17</sup> Formally, we consider the following two-stage game.

Fix an initial network  $g$  connecting agents. In the first stage, each agent  $i = 1, \dots, n$  decides to enter the labor market or to become a delinquent. This is a simple binary decision. These decisions are simultaneous. Let  $c_i \in \{0, 1\}$  denote  $i$ 's decision, where  $c_i = 1$  (respectively,  $c_i = 0$ ) stands for becoming a delinquent (respectively, entering the labor market), and denote by  $\mathbf{c} = (c_1, \dots, c_n)$  the corresponding population binary decision profile. We assume that agents entering the labor market earn a fixed wage (nonnegative scalar)  $\omega > 0$ . The payoff for delinquents is determined in the second stage of the game. In the second stage, delinquents decide their effort level, which depends on the first-stage outcome.

**DEFINITION 3.** The *extended game* is a two stage game where in stage 1, each player  $i \in N$  decides whether to participate ( $c_i = 1$ ) or not ( $c_i = 0$ ) to the crime market, and in stage 2, the players play the game in  $g_S$ ;<sup>18</sup> The utilities are given by

$$U_i(S, \mathbf{x}_S, g) = \begin{cases} u_i(\mathbf{x}_S, g_S) & \text{if } i \in S, \\ \omega & \text{otherwise.} \end{cases}$$

We study the *subgame perfect equilibrium in pure strategies* of this extended game.

**DEFINITION 4.** The set  $S$  is supported in equilibrium if there exists a  $\omega$  and a subgame perfect equilibrium where the set of players who decide to participate is  $S$ , given the outside option  $\omega$ .  $S$  is also called an (equilibrium) participation pool of the game at the wage level  $\omega$ .

17. See Calvó-Armengol and Jackson (2004) for a similar endogenous game of network formation in the context of the labor market, where the binary decision for agents is to enter the labor market network or to drop out.

18.  $S$  is the set of players who decide to participate.

Let  $\mathcal{E}(\omega)$  be the family of sets supported by  $\omega$  at equilibrium in the extended game. The following result characterizes the class of sets that can be supported by some  $\omega$ .

**PROPOSITION 8.** *Let  $S \subseteq N$  and  $\theta\rho(g) < 1$  for all  $j \in N \setminus S$ . Then, the set  $S$  is supported at equilibrium by the outside option  $\omega$  if and only if*

$$\max_{j \in N \setminus S} \frac{b_j(g_{S \cup \{j\}}, \theta)}{1 + b(g_{S \cup \{j\}}, \theta)} \leq \frac{1}{1 - \pi} \sqrt{\omega\delta} \leq \min_{i \in S} \frac{b_i(g_S, \theta)}{1 + b(g_S, \theta)}.$$

See Ballester, Calvó-Armengol, and Zenou (2009) for the proof.

**REMARK 2.** Whenever

$$\omega > \frac{(1 - \pi)^2}{4\delta},$$

all agents outside the delinquency pool is an equilibrium, that is,  $\emptyset$  is supported as an equilibrium by  $\omega$ .

Whenever an equilibrium exists, multiplicity of equilibria is a natural outcome of the extensive form game. This multiplicity can arise, for instance, from the symmetric role of some agents in a network.<sup>19</sup>

#### 4.2. Participation Game without Global Substitutability

Suppose that  $\delta$  is small, that is, we have that the second-stage game is close to a game with strategic complementarities. Let  $S$  be a participation pool (not necessarily an equilibrium pool) at some wage  $\omega$ . In this case, the payoff that an agent  $i \in N \setminus S$  obtains by joining  $S$  is equal to

$$x_i^*(g_{S \cup \{i\}}) = \left( \frac{1 - \pi}{\delta} \right) b_i(g_{S \cup \{i\}}, \theta),$$

that is, it is proportional to its centrality in the network  $g_{S \cup \{i\}}$ .

Given that the outside option  $\omega$  is fixed, it is clear that the two-stage game is *supermodular*,<sup>20</sup> in the sense that the payoffs of player  $i$  are increasing with

19. Two agents  $i$  and  $j$  are *symmetric* in a network whenever the network remains with the same structure after exchanging their labels. In this case, if  $S$  is supported at equilibrium,  $i \in S$  and  $j \in N \setminus S$ , so is  $S'$  where  $i$  has been interchanged with  $j$ .

20. For a broad description of supermodular games and their applications, see Topkis (1998) and Vives (2005).

respect to participation decisions of other agents. Formally, for all  $S \subseteq T \subseteq N$  and  $i \in N \setminus T$ , it is clear that

$$b_i(g_{S \cup \{i\}}, \theta) \leq b_i(g_{T \cup \{i\}}, \theta)$$

because the right-hand side measures a higher number of walks.

This property ensures the existence of equilibrium for any wage  $\omega$ , as summarized by the following proposition.

**PROPOSITION 9.** *When  $\delta$  is small, the extended game has at least one equilibrium participation pool.*

See Ballester, Calvó-Armengol, and Zenou (2009) for the proof.

One may be interested in providing all the possible equilibria of the game when supermodularity holds. Echenique (2007) provides a useful tool to list all the equilibria of a game with complementarities. The intuitive idea here is that complementarity is low enough to allow for increasing differences in utility of agents in their decisions to enter the participation pool.

### 4.3. Finding the Key Player with Criminal Participation Decision

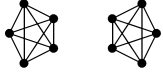

Given that this game usually displays multiple subgame perfect equilibria in the endogenous delinquency network game, we define  $x^*(g, \omega)$  to be the maximum aggregate equilibrium delinquency level when the delinquency network is  $g$  and the labor market wage is  $\omega$ . This delinquency level is equal to the total amount of delinquency in the worst case scenario of maximum delinquency.

Consider some binary decision profile  $\mathbf{c}$ . Let  $i$  be an active delinquent, that is,  $c_i = 1$ . Suppose that delinquent  $i$  switches his current decision to  $c_i = 0$ , that is, delinquent  $i$  drops out from the delinquency pool and enters the labor market instead. The binary decision profile then becomes  $\mathbf{c} - \mathbf{v}^i$ , and the new set of active delinquents is  $C(\mathbf{c} - \mathbf{v}^i) = C(\mathbf{c}) \setminus \{i\}$ . The drop out of delinquent  $i$  from the delinquency pool also alters the network structure connecting active delinquents, as any existing direct link between  $i$  and any other delinquent in  $C(\mathbf{c})$  is removed. The new network connecting active delinquents is then  $g(\mathbf{c})^{-i} = g(\mathbf{c} - \mathbf{v}^i)$ , and the aggregate delinquency level becomes

$$x^*(\mathbf{c} - \mathbf{v}^i) = \frac{1 - \pi}{\delta} \frac{b(g(\mathbf{c} - \mathbf{v}^i), \theta)}{1 + b(g(\mathbf{c} - \mathbf{v}^i), \theta)}.$$

The key player problem acquires a different shape in the setting with endogenous formation of delinquency pools. Initially, the planner must choose a player to remove from the network. Then, players play the two-stage delinquency game.

TABLE 4. Key player and criminal decision.

	$\omega = 0.001$	$\omega = 0.003$
$x^*(g_{-1}, \omega)$	0.7843	0.7843
$x^*(g_{-2}, \omega)$	0.7847	0.7785
Key player	1	2
Final delinquency pool		

First, they decide whether to enter the delinquency pool or not. Second, delinquents choose how much effort to exert. In this context, there is an added difficulty to the planner’s decision. The removal of a player from the network affects the rest of the players’ decisions to become active delinquents. This fact should be taken into account by the planner in order to attain an equilibrium with minimum total delinquency. The right choice of the key player should be based upon the remaining delinquency pool that will result from that decision, that is, what the remaining players will decide concerning their delinquent activities.

We show, with the help of an example, that there is no trivial geometric recipe for the key player problem in this case.

Consider again the network in Figure 1 with 11 players. Recall that, when  $\theta = 0.2$  and the network of delinquents is exogenously fixed (or, equivalently, the outside option is  $\omega = 0$ ), the key player was the player acting as a bridge, namely, delinquent 1. If we now consider the endogenous delinquency network formation in the two-stage game, the results may differ. Indeed, for low wages, player 1 is also the key player and the resulting equilibrium network is the whole remaining network, that is, ten delinquents remain and are split into two fully connected cliques of five delinquents. However, when  $\omega$  becomes higher, delinquent 2 becomes the key player<sup>21</sup> and the equilibrium network now encompasses six different players. It consists of a clique of five fully intraconnected players together with player 1.

These results are summarized in Table 4, which gives, for two different values of  $\omega$ , the key player, the highest aggregate delinquency that results from eliminating this key player, and the equilibrium delinquency network.

Intuitively, when outside opportunities are high enough, all players from the same side of the player being removed do not have enough incentives to enter the delinquency pool at the first stage of the game. Hence, we do not get a “large” equilibrium with many players, and this constitutes an advantage for the planner who will choose to delete player 2. This example implicitly explains how one policy (providing a higher  $\omega$ ) increases the effectiveness of another policy (choosing the key player) in order to reduce delinquency. These policies are complementary from the point of view of their effects on total delinquency,

21. In fact, any player except player 1 is the key player for  $\omega = 0.003$ .



although we are aware that they may be substitute if we had considered a budget-restricted planner who had to implement costly policies.

#### 4.4. The Key-Group Problem with Criminal Participation Decision

In the simple case without outside option, the choice of the key group was based on the contribution of that group to the connectivity (total Katz–Bonacich centrality) of the network. In the context of games with criminal participation decision, an additional criterion should be taken into account: the fact that a removal of some players may induce further voluntary moves of other players in the network. Thus, the choice of the optimal target can change accordingly, and differ from the usual key group prescription when all players participate. We analyze the interplay between the optimal target and an outside option that acts as a participation threshold in the new game.

The issue of existence will be relaxed in this section by assuming that we are dealing with a wage  $\omega$  such that, for any subgame in the subnetwork  $g_T$ , with  $T \subseteq N$ , there exists an equilibrium participation pool  $S$  supported by  $\omega$ . On the other hand, multiplicity of equilibria makes it difficult to adopt a particular approach in order to assess the efficacy of a particular policy. In this paper, we focus on a extreme approach where the removal of a set of players from the network is evaluated by comparing the maximum equilibrium of the original game and the resulting game, that is, outcomes with maximum total activity  $x^*$ .

**DEFINITION 5.** Given an extended game with wage  $\omega$  and  $T \subseteq N$ , the remaining family after eliminating  $S$  is defined as

$$P(\omega, S) = \{T \subseteq N \setminus S, T \in \mathcal{E}(\omega)\}.$$

In words, a set  $T \subseteq N \setminus S$  is in the remaining family after eliminating  $S$  whenever  $T$  is a participating pool in the restricted extended game played in  $g_{-S}$ . This definition is just capturing the posterior behavior of players after  $S$ 's removal.

For a candidate set  $S$  to be eliminated, let  $P_m(\omega, S) \in \arg \max_{T \in P(\omega, S)} \{b(g_T, \theta)\}$  be a *maximum equilibrium* participation pool when the set  $S$  is eliminated. It is a pool where the maximum activity is achieved. Then, the choice of a key group  $S^*$  of size  $s$  is

$$S^* \in \arg \min_{|S| \leq s} b(g_{P_m(\omega, S)}, \theta).$$

Let us illustrate this with the network described in Figure 1. We study the problem of eliminating one player ( $s = 1$ ) when  $\theta = 0.2$ . When we analyze the extended game with criminal participation decision, it is crucial for the planner to consider the possible transitions between different pools of delinquents. In

TABLE 5. Group players and criminal decision.

Pool $S$	$g_{-1}$			$g_{-2}$		
	$\omega_L$	$\omega_H$	$b(g_S, \theta)$	$\omega_L$	$\omega_H$	$b(g_S, \theta)$
{2, 3, 4, 5, 6, 7, 8, 9, 10, 11}	0.00	12.50	50.00	—	—	—
{2, 3, 4, 5, 6}	0.50	12.50	25.00	—	—	—
{1, 6, 7, 8, 9, 10, 11}	—	—	—	1.02	1.70	39.47
{1, 3, 4, 5, 6, 7, 8, 9, 10, 11}	—	—	—	0.00	4.17	51.33
{1, 7, 8, 9, 10, 11}	—	—	—	1.70	7.03	36.25
{7, 8, 9, 10, 11}	—	—	—	7.03	12.50	25.00
{1, 3, 4, 5, 6}	—	—	—	0.95	1.25	12.37
{3, 4, 5, 6}	—	—	—	1.25	3.12	10.00

particular, there are now three effects that should be taken into account when choosing the set of players to be removed:

1. A *direct effect* due to the reduction of their initial delinquent activity. The choice is here biased towards the most Katz–Bonacich central players.
2. An *indirect effect* due to the (lower) incentives of the remaining players. In this dimension, group-intercentrality is the relevant variable to consider.
3. A possible *snowball effect* because the removal of a player may induce a process where the remaining players (sequentially) find it profitable to leave the pool of delinquents and to participate in the labor market. This effect depends on the magnitude of the outside option  $\omega$ .

Table 5 summarizes the sets  $S$  that are sustainable in equilibrium for the games played in  $g_{-1}$  (i.e., when delinquent 1 is removed) and  $g_{-2}$  (i.e., when delinquent 2 is removed), specifying the range  $[\omega_L, \omega_H]$  of wages that support for each  $S$  an equilibrium participating pool (we can have multiplicity of equilibria). We just specify distinct (up to network isomorphism) equilibrium pools.

To illustrate the results given in this table, let us consider the case when  $\omega = 5$ . If delinquent 1 is removed then the highest equilibrium pool consists of all delinquents but 1, that is  $N \setminus \{1\} = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ . This is exactly the same choice as in the case without criminal participation decision, which is based on the intercentrality index. If, on the contrary, delinquent 2 is removed then for the wage  $\omega = 5$ , it is easily seen that the maximum equilibrium pool becomes  $\{1, 7, 8, 9, 10, 11\}$ . In other words, with the exception of 1, all the delinquents directly connected to delinquent 2 find it not profitable to become delinquents and instead prefer to participate in the labor market. As a result, when the wage is  $\omega = 5$  and there is no criminal participation decision, then the key player is delinquent 2 because its deletion from the network has the highest impact on the incentives of other players to become delinquent. In other words, by deleting delinquent 2 instead of delinquent 1, fewer individuals will become delinquent. This will also lead to a larger decrease in the aggregate level of crime.

If we now consider a much lower wage, say  $\omega = 0.4$ , then removing delinquent 1 or 2 will have the same effect on individuals' participation in criminal activities. Indeed, in both cases, all individuals will find it profitable to be delinquent because when delinquent 1 is removed,  $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$  will be criminals in equilibrium, whereas, when delinquent 2 is removed, the equilibrium pool of delinquents is  $\{1, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ . This suggests that the effectiveness of a key-player policy should not only be measured by the direct and indirect effects on delinquent activities but also by the group interactions it engenders in terms of participation to the labor market.

## References

- Akers, Ronald (1998). *Social Learning and Social Structure: A General Theory of Crime and Deviance*. Northeastern University Press.
- Baker, Wayne, and Robert Faulkner (1993). "The Social Organization of Conspiracy: Illegal Networks in the Heavy Electrical Equipment Industry." *American Sociological Review*, 58, 837–860.
- Ballester, Coralio (2004). "NP-Completeness in Hedonic Games." *Games and Economic Behavior*, 49, 1–30.
- Ballester, Coralio, Antoni Calvó-Armengol, and Yves Zenou (2006). "Who's Who in Networks. Wanted: The Key Player." *Econometrica*, 74, 1403–1417.
- Ballester, Coralio, Antoni Calvó-Armengol, and Yves Zenou (2009). "Delinquent Networks." CEPR Discussion Paper No. 7293.
- Bayer, Patrick, Randi Hjalmarsson, and David Pozen (2009). "Building Criminal Capital Behind Bars: Peer Effects in Juvenile Corrections." *Quarterly Journal of Economics*, 124, 105–147.
- Becker, Gary (1968). "Crime and Punishment: An Economic Approach." *Journal of Political Economy*, 76, 169–217.
- Bonacich, Phillip (1987). "Power and Centrality: A Family of Measures." *American Journal of Sociology*, 92, 1170–1182.
- Burdett, Kenneth, Ricardo Lagos, and Randall Wright (2003). "Crime, Inequality and Unemployment." *American Economic Review*, 93, 1764–1777.
- Calvó-Armengol, Antoni, and Matthew Jackson (2004). "The Effects of Social Networks on Employment and Inequality." *American Economic Review*, 94, 426–454.
- Calvó-Armengol, Antoni, Eleonora Patacchini, and Yves Zenou (2005). "Peer Effects and Social Networks in Education and Crime." CEPR Discussion Paper No. 5244.
- Calvó-Armengol, Antoni, Eleonora Patacchini, and Yves Zenou (2009). "Peer Effects and Social Networks in Education." *Review of Economic Studies*, 76, 1239–1267.
- Calvó-Armengol, Antoni, and Yves Zenou (2004). "Social Networks and Crime Decisions: The Role of Social Structure in Facilitating Delinquent Behavior." *International Economic Review*, 45, 935–954.
- Costebander, Elizabeth, and Thomas Valente (2003). "The Stability of Centrality Measures when Networks are Sampled." *Social Networks*, 25, 283–307.
- Damm, Anna, and Christian Dustmann (2008). "Do Young People Learn Criminal Behavior? Quasi-Experimental Evidence." Working paper, University College London.
- Echenique, Frederico (2007). "Finding All Equilibria in Games with Strategic Complements." *Journal of Economic Theory*, 135, 514–532.
- Garey, Michael, and David Johnson (1978). *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman.

- Garoupa, Nuno (1997). "The Theory of Optimal Law Enforcement." *Journal of Economic Surveys*, 11, 267–295.
- Glaeser, Edward, Bruce Sacerdote, and Jose Scheinkman (1996). "Crime and Social Interactions." *Quarterly Journal of Economics*, 111, 508–548.
- Glaeser, Edward, Bruce Sacerdote, and Jose Scheinkman (2003). "The Social Multiplier." *Journal of the European Economic Association*, 1, 345–353.
- Goyal, Sanjeev (2007). *Connections: An Introduction to the Economics of Networks*. Princeton University Press.
- Haynie, Dana (2001). "Delinquent Peers Revisited: Does Network Structure Matter?" *American Journal of Sociology*, 106, 1013–1057.
- Jackson, Matthew (2008). *Social and Economic Networks*. Princeton University Press.
- Jacob, Brian, and Lars Lefgren (2003). "Are Idle Hands the Devil's Workshop? Incapacitation, Concentration, and Juvenile Crime." *American Economic Review*, 93, 1560–1577.
- Katz, Leo (1953). "A New Status Index Derived from Sociometric Analysis." *Psychometrika*, 18, 39–43.
- Ludwig, Jens, Greg Duncan, and Paul Hirschfield (2001). "Urban Poverty and Juvenile Crime: Evidence from a Randomized Housing-Mobility Experiment." *Quarterly Journal of Economics*, 116, 655–679.
- Matsueda, Ross L., and Kathleen Anderson (1998). "The Dynamics of Delinquent Peers and Delinquent Behavior." *Criminology*, 36, 269–308.
- Nemhauser, George, Laurence Wolsey, and Marshall Fisher (1978). "An Analysis of Approximations for Maximizing Submodular Set Functions—I." *Mathematical Programming*, 14, 265–294.
- Patacchini, Eleonora, and Yves Zenou (2008). "The Strength of Weak Ties in Crime." *European Economic Review*, 52, 209–236.
- Polinsky, Mitchell, and Steven Shavell (2000). "The Economic Theory of Public Enforcement of Law." *Journal of Economic Literature*, 38, 45–76.
- Sarnecki, Jerzy (2001). *Delinquent Networks: Youth Co-Offending in Stockholm*. Cambridge University Press.
- Sutherland, Edwin (1947). *Principles of Criminology*. 4th edition. J. B. Lippincott.
- Topkis, Donald (1998). *Submodularity and Complementarity*. Princeton University Press.
- Verdier, Thierry, and Yves Zenou (2004). "Racial Beliefs, Location and the Causes of Crime." *International Economic Review*, 45, 727–756.
- Vives, Xavier (2005). "Complementarities and Games: New Developments." *Journal of Economic Literature*, 43, 437–479.
- Warr, Mark (1966). "Organization and Instigation in Delinquent Groups." *Criminology*, 34, 11–37.
- Warr, Mark (2002). *Companions in Crime: The Social Aspects of Criminal Conduct*. Cambridge University Press.